From Triviality to a nontrivial QFT

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(joint work with Raimar Wulkenhaar)

Introduction

Motivation: Improve local QFT, add quantum gravity effects.
There are no nontrivial local QFTs in 4 D constructed (obeying the Axioms).
Candidates are:

- perturbative renormalizable fields, Φ^4 (UV? Landau ghost).
- YM (asymptotic free) IR not understood (confinement, renormalons).

We show: 4D-ϕ^4-model can be constructed on a NC MF.

- β function vanishes, model asymptotic safe.
- Constructed on Euclidean Space-Time, model solved.
- Infinitely many renormalisable Feynman graphs summed up.
- Analytic continuation under study.

Minkowski ST

- Mixing occurs: D. Bahns.

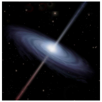
Two Pillars:

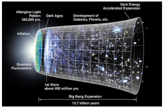
QUANTUM physics
Quantize Phase Space
$\{f, g\} \to [\hat{f}, \hat{g}]$

- $[\hat{x}, \hat{p}] = i\hbar$
- Ritz $\nu_{i,j} = \nu_i - \nu_j$
  $\nu_{i,j} = \nu_{i,k} + \nu_{k,j}$
  $x_{i,j}(t) = x_{i,j}(0)e^{2\pi i \nu_{i,j} t}$
  gives matrix product
- $i\hbar \frac{d}{dt}\hat{x} = [\hat{H}, \hat{x}]$ is derivation
  example of nc geometry

generalize to QFT (Axioms)
renormalizable QFT

classical GRAVITY
Riemannian geometry





- Riemann curvature
- Einstein Equations
  $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$
- G has dimension:
  Gravity is not renormalizable

Attempts:
Strings, SUGRA, LQG, NCQFT

Canonical quantization: incompatible

Lattice Models

To each lattice point $x = (x_1, \ldots, x_D)$ assign variable ϕ_x. The distribution of a spin given by positive measure $d\mu(\phi_x)$
Interaction is ferromagnetic:

$$H_V(\phi) = -\sum_{\langle xy \rangle} \phi_x \cdot \phi_y - h \cdot \sum_x \phi_x,$$

ϕ fulfills periodic boundary conditions.

$$\langle \mathcal{F} \rangle_{V,a} = \frac{1}{Z_V} \int \prod_{x \in V} d\mu(\phi_x) \mathcal{F}(\phi)\, e^{-\beta H_V(\phi)},$$

$d\mu(\vec{\phi}) = \delta(|\vec{\phi}|^2 - 1) d^n\phi$ is Ising model for $n = 1$ and Heisenberg model for $n = 3$.
Measure $d\mu(\phi) = \exp(-\lambda\phi^4 - \mu^2\phi^2) d\phi$ gives lattice regularization of the ϕ^4 model.

$$\langle \Phi_i \Phi_j \rangle^c \equiv \langle \Phi_i \Phi_j \rangle - \langle \Phi_i \rangle \langle \Phi_j \rangle \simeq \text{const} \frac{\exp(-\frac{|i-j|}{\xi})}{|i-j|^{D-2}} \qquad |i-j| \to \infty, T \neq T_c.$$

$$\langle \Phi_i \Phi_j \rangle^c \simeq \frac{\text{const}}{|i-j|^{D-2+\eta}} \qquad \text{for } |i-j| \to \infty, \qquad T = T_c.$$

Reflection Positivity

Take finite lattice, symmetric wrt a time zero plane lying half a lattice distance away from the lattice points.

Divide Λ into Λ_+ and Λ_-, of equal size, and $\Lambda_+ \cup \Lambda_- = \Lambda$.

Denote by $\mathcal{A}_+$ (or $\mathcal{A}_-$) the algebra of observables (functions) which are supported on Λ_+ (or Λ_-).

Define an antilinear map Θ from $\mathcal{A}_+$ to $\mathcal{A}_-$ by

$$\mathcal{A}_+ \stackrel{\Theta}{\to} \mathcal{A}_-, \qquad f(\phi_x) \stackrel{\Theta}{\to} f^*(\phi_{rx})$$

with $rx = (-x_1, x_2, ..., x_d)$ if $x = (x_1, ..., x_d)$, * is cc

Θ is cc and reflection at the $t = 0$ plane, which is not a lattice plane.

Reflection positivity property:

$\langle \Theta A \cdot A \rangle \geq 0 \qquad \forall A \in \mathcal{A}_+$ Fulfilled for Φ^4 model.

Gives $|\int d\mu(x) d\mu(y) F^*(x) G(y)\, e^{-\frac{\beta}{2}(x-y+h)^2}|^2 \leq ||F||_\beta^2 ||G||_\beta^2$,

$$||F||_\beta^2 = |\int d\mu(x) d\mu(y) F^*(x) F(y)\, e^{-\frac{\beta}{2}(x-y)^2}|$$

For $h_\alpha = (h_1, ..., h_d)$ with values in $\mathbf{R}^d$, it follows with $\Phi(h) = \sum_x f(x)\Phi_x$ and $(\partial_\alpha h)(x) = h(x + e_\alpha) - h(x)$

Triviality

$$\langle \exp(\sum_{\alpha=1}^{D} \phi(\partial_\alpha h_\alpha))\rangle \le \exp(\frac{1}{2\beta}\sum_\alpha ||h_\alpha||_2^2), \qquad ||h||_2^2 = \sum_x |h(x)|^2.$$

Existence of phase transitions follow from infrared bounds:
$\hat{G}(\vec{k}) = M^2\delta^D(\vec{k}) + \rho(\vec{k})$ with $0 \le \rho(\vec{k}) \le \frac{\text{const}}{\beta\cdot|\vec{k}|^2}$.
For the renormalized field it follows that

$$0 < G_\Theta(x-y) = Z(\Theta)\langle \phi_{\Theta x}\phi_{\Theta y}\rangle \le \frac{Z(\Theta)\cdot \text{const}}{\beta(\Theta)\cdot\Theta^{D-2}|x-y|^{D-2}}.$$

implies: $Z(\Theta) \ge \Theta^{D-2}$ Correlation inequality (random walk repr.):

$$Z^2(\Theta)G_4^c(\Theta x_1,\ldots) \ge -\beta^2 Z^2(\Theta)\sum_z \langle\phi_{\Theta x_1}\phi_z\rangle\langle\phi_z\phi_{\Theta x_2}\rangle\langle\phi_{\Theta x_3}\phi_z\rangle\langle\phi_z\phi_{\Theta x_4}\rangle,$$

implies: $\Theta^D Z^{-2}(\Theta) \le \Theta^{4-D}$. [Aizenman, Fröhlich]
Implies that $G_4^c \to 0$ for $\Theta \to \infty$ and $D > 4$.
The model has a trivial continuum limit for $D > 4$.

Requirements

Quantum mechanical properties

- states are vectors of a separable Hilbert space H
- $\Phi(f)$ on D dense, $\Phi(f) = \int d^4x \Phi(x) f^*(x)$, Ω is cyclic
- Space-time translations are symmetries:
  spectrum $\sigma(P_\mu)$ in closed forward light cone
  Ground state $\Omega \in H$ invariant under $e^{ia_\mu P^\mu}$

Relativistic properties

- $U_{(a,\Lambda)}$ unitary rep. of Poincaré group on H, Covariance
- Locality: $[\phi(f), \phi(g)]\Psi = 0$ for $suppf \subset (suppg)'$

Define Wighman functions $W_N(f_1 \otimes ... \otimes f_N) :=< \Omega \phi(f_1) \cdots \phi(f_N) \Omega >$ Define Schwinger functions $S_N(z_1, ..., z_N) = \int \Phi(z_1)...\Phi(z_N) d\nu(\Phi)$

ϕ is a stochastic variable, define measure

euclidean invariant, symmetric (from microcausality),
fulfills reflection positivity, cluster property,...

Φ^4 Interaction

Let $\langle\phi(x_1)\phi(x_2)\rangle = C(x_1, x_2)$ be free 2-pt fct, expand $e^{\int dx \Phi^4}$
Feynman perturbation expansion: Regularize, renormalize

$$S_{N,V,\Lambda}(x_1 \dots x_N) = \sum_n \frac{(-\lambda)^n}{n!} \int d\mu_\Lambda(\phi) \prod_j^N \phi(x_j) \left(\int_V dx \, \frac{\phi^4(x)}{4!} \right)^n$$

$$= \sum_{\text{graph } \Gamma_N} \frac{(-\lambda)^n}{Sym_{\Gamma_N}(G)} \int_V \prod_{l \in \Gamma_N} C_\Lambda(x_l - y_l) \sim \Lambda^{\omega_D(G)}$$

put cutoffs, e.g.: $\tilde{C}_\Lambda(p) = \int_{1/\Lambda^2}^\infty d\alpha e^{-\alpha(p^2+m^2)}$

degree of divergence given by $\omega_D(G) = (D-4)n + D - \frac{D-2}{2}N$

n order, $\#$ of vertices, N, $\#$ of external lines,

BPHZ theorem, require 3 normalisation conditions, follow RG-flow

Expansions not summable!

RG FLOW

Wilson RG-Flow

divide covariance for free Euclidean scalar field into slices

$$\Phi_m = \sum_{j=0}^{m} \phi_j, \quad C_j = \int_{M^{-2j}}^{M^{-2(j-1)}} d\alpha \frac{e^{-m^2\alpha - x^2/4\alpha}}{\alpha^{D/2}}$$

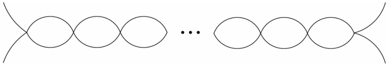
integrate out degrees of freedom

$$Z_{m-1}(\Phi_{m-1}) = \int d\mu_m(\phi_m) e^{-S_m(\phi_m + \Phi_{m-1})} = e^{-S_{m-1}(\Phi_{m-1})}$$

$$\lambda_m \simeq \frac{\lambda_0}{1-\beta\lambda_0 m}$$

to evaluate: use loop expansion

certain chain of finite subgraphs with p bubbles grows like



$$\int \frac{d^4q}{(q^2+m^2)^3} (\log|q|)^p \simeq c^p p!,$$

sign of β positive: Landau ghost, triviality

Ideas

Limited localization of events in space-time

$$D \geq R_{ss} = G/c^4 hc/\lambda \geq G/c^4 hc/D$$

gives Planck lenght as a lower limit to localization

Riemann, Schrödinger, Heisenberg, Peierls, Pauli, Oppenheimer, Snyder, ... Bronstein, Karolyhazy, Mead, Kempf,...

1986 Connes: Noncommutative Geometry
1995 Filk: Feynman rules, DFR: uncertainty rel...
1999 Schomerus: obtains nc models from strings

- Replace manifold by algebra, deform it
- keep differential calculus derivations, covariant derivative,...
- Replace fields by projective modules
- Replace integrals by traces
- use renormalized perturbation expansion, extend to nc geometry

Replace de Rham cohomology by cyclic cohomology
study spectral triples, spectral action principle

Can we make sense out of renormalisation in NCG?

Construct QFT on simple nc geometries, e.g. Weyl algebra

$$W(f) = \int dp e^{ip\hat{x}} \int dx e^{-ipx} f(x)$$

with $[\hat{x}_\mu, \hat{x}_\nu] = i\Theta_{\mu\nu}$, $\qquad W(f)W(g) = W(f \star g)$

Moyal space

algebra of rapidly decaying fcts on D-dimensional Eucl. space with $\star$-product

$$(a \star b)(x) = \int d^D y d^D k a(x + \tfrac{1}{2}\Theta \cdot k) b(x + y) \, e^{iky}$$

where $\Theta = -\Theta^T \in M_D(\mathbb{R})$

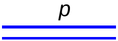
- $\star$-product is associative, noncommutative, and: non-local
- construction of field theories with non-local interaction

ϕ^4-action

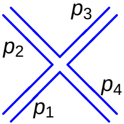
- naïve ϕ^4-action (ϕ-real, Euclidean space) on Moyal plane

$$S = \int d^4x \left(\frac{Z_\Lambda}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{Z_\Lambda m^2}{2} \phi \star \phi + \frac{Z_\Lambda^2 \lambda}{4} \phi \star \phi \star \phi \star \phi \right)(x)$$

- Feynman rules:



$$= \frac{1}{p^2 + m^2}$$

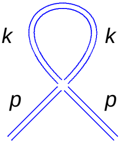


$$= \frac{\lambda}{4} \exp\left(-\frac{\mathrm{i}}{2} \sum_{i<j} p_i^\mu p_j^\nu \theta_{\mu\nu} \right)$$

- cyclic order of vertex momenta is essential $\Rightarrow$ ribbon graphs

Mixing

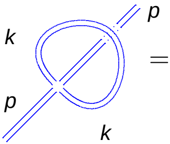
- one-loop two-point function, *planar contribution*:



$$= \frac{\lambda}{6}\int^{\Lambda} \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}$$

to be treated by usual regularisation methods,

- planar nonregular, nonplanar graphs finite (nc as regulator)



$$= \frac{\lambda}{12}\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot\Theta\cdot p}}{k^2+m^2} \sim (\Theta p)^{-2}$$

- but $\sim p^{-2}$ for small momenta (renormalisation not possible).

$\Rightarrow$ leads to non-integrable integrals when inserted as subgraph into bigger graphs: UV/IR-mixing: NONRENORMALIZABLE

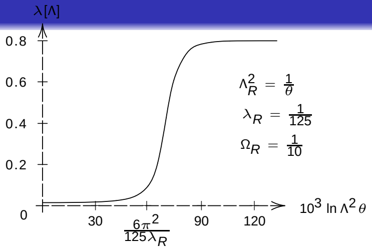
ϕ^4_4 on Moyal space with harmonic propagation

ϕ^4-theory on 4*D*-Moyal space with harmonic oscillator potential

$$S[\phi] = \int d^4x \Big(\frac{Z_\Lambda}{2}\phi \star (-\Delta + \Omega^2(2\Theta^{-1}x)^2 + \mu_{bare}^2)\phi + \frac{\lambda Z_\Lambda^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

- renormalizable as formal power series in λ [HG-R. Wulkenhaar] (renormalisation of $\mu_{bare}^2, \lambda, Z_\Lambda \in \mathbb{R}_+$ and $\Omega \in (0,1]$) means: well-defined perturbative quantum field theory
- Langmann-Szabo duality: theories at Ω and $\Omega^* = \frac{1}{\Omega}$ are the same; self-dual case $\Omega = 1$ is matrix model
- β-function vanishes to all orders in λ for $\Omega = 1$ [Disertori-Gurau-Magnen-Rivasseau], almost scale-invariant
- added term is curvature term
- Model constructed and solved [HG + R. Wulkenhaar]

The β-function



one-loop calculation

$$\Lambda \frac{d\Omega}{d\Lambda} = \beta_\Omega = \Omega\lambda \frac{(1-\Omega^2)}{(1+\Omega^2)^3} + \mathcal{O}(\lambda^3)$$

$$\Lambda \frac{d\lambda}{d\Lambda} = \beta_\lambda = 2\lambda^2 \frac{(1-\Omega^2)}{(1+\Omega^2)^3} + \mathcal{O}(\lambda^3)$$

$\lambda[\Lambda]$ diverges in commutative case

- perturbation theory remains valid at all scales!
- Non-perturbative construction: Finite Renormalization!

How does this work?

- four-point function renormalisation with usual sign
- $\exists$ one-loop wavefunction renormalisation which compensates four-point function renormalisation for $\Omega \to 1$

Landau Ghost tamed!

Ward identity

Go to Matrix Base $\quad H_{nm} = n + m + \mu^2$

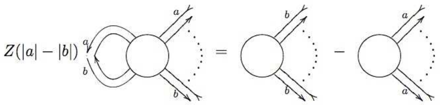
$$S[\Phi] = \sum_{n,m} \frac{Z_\Lambda}{2} \Phi_{nm} H_{nm} \Phi_{mn} + \frac{\lambda}{4} \sum_{nmpq} Z_\Lambda^2 \Phi_{nm} \Phi_{mp} \Phi_{pq} \Phi_{qn}$$

inner automorphism $\phi \mapsto U\phi U^\dagger$ of M_Λ, infinitesimally,
not a symmetry of the action, but invariance of measure

Interpretation

Insertion of special vertex $V_{ab}^{ins} := Z_\Lambda \sum_n (H_{an} - H_{nb}) \phi_{bn} \phi_{na}$

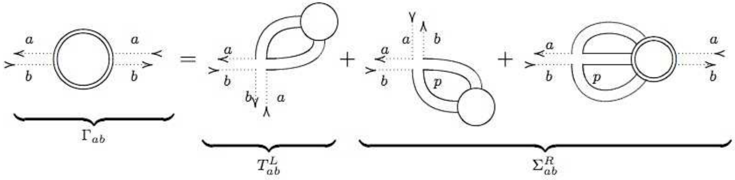
into external face equals the difference between the exchanges of external sources $J_{nb} \mapsto J_{na}$ and $J_{an} \mapsto J_{bn}$



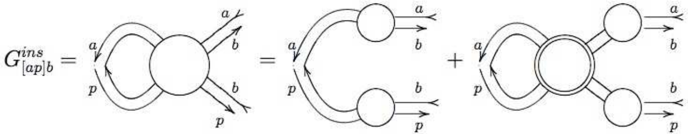
The dots stand for the remaining face indices.

$$Z_\Lambda(|a| - |b|) G^{ins}_{[ab]\ldots} = G_{b\ldots} - G_{a\ldots}$$

SD equation 2



- vertex is $Z_\Lambda^2\lambda$, connected two-point function is G_{ab}: first graph equals $Z_\Lambda^2\lambda\sum_q G_{aq}$
- open p-face in Σ^R and compare with insertion into connected two-point function



gives for 2 point function:

$$Z_\Lambda^2\lambda\sum_q G_{aq} - Z_\Lambda\lambda\sum_p (G_{ab})^{-1}\frac{G_{bp}-G_{ba}}{|p|-|a|} = H_{ab} - G_{ab}^{-1}\,.$$

Result

renormalize, remove cutoffs, gives singular integral equation, solve,...

Theorem

$$\frac{(1-\beta)}{1-\alpha\beta}\frac{G_{\alpha\beta}}{1+\lambda\mathcal{Y}} = \frac{\sin(\theta_\beta(\alpha))}{|\lambda|\pi\alpha} e^{\mathcal{H}_\alpha[\theta_\beta(\bullet)] - \mathcal{H}_0[\theta_0(\bullet)] + \mathcal{H}_1[\theta_0(\bullet) - \theta_\beta(\bullet)]}$$

$$\frac{\lambda\mathcal{Y}}{1+\lambda\mathcal{Y}} = \int_0^1 d\rho \, \frac{\sin^2(\theta_0(\rho))}{\lambda\pi^2\rho^2}$$

$$\theta_\beta(\alpha) = \operatorname*{arctan}_{[0,\pi]} \left(\frac{\lambda\pi\alpha}{\frac{\beta(1-\alpha)}{1-\beta} + \frac{1+\lambda\mathcal{Y}+\lambda\pi\alpha\mathcal{H}_\alpha[G_{\bullet 0}]}{G_{\alpha 0}}} \right) \qquad (*)$$

Consequence: $G_{\alpha\beta} \geq 0$!

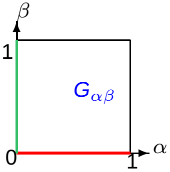
Main steps of the proof:

1. (*) is Carleman eq. $\lambda\pi \cot\theta_0(\alpha) G_{\alpha 0} - \lambda\pi\mathcal{H}_\alpha[G_{\bullet 0}] = \frac{1+\lambda\mathcal{Y}}{\alpha}$
2. Tricomi's identity $e^{-\mathcal{H}_\alpha[\theta_\beta]}\cos(\theta_\beta(\alpha)) + \mathcal{H}_\alpha\left[e^{-\mathcal{H}_\bullet[\theta_\beta]}\sin(\theta_\beta(\bullet)\right] = 1$

The self-consistency equation

Given boundary value $G_{\alpha 0}$,
Carleman computes $G_{\alpha\beta}$,
in particular $G_{0\beta}$

symmetry forces $G_{\beta 0} = G_{0\beta}$



Master equation

The theory is completely determined by the solution of

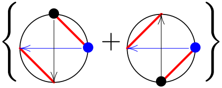
$$G_{\beta 0} = \frac{1+\lambda\mathcal{Y}}{1+(1-\beta)\lambda\mathcal{Y}} \times \exp\left(-\lambda \int_0^{\frac{\beta}{1-\beta}} dt \int_0^1 \frac{d\rho}{(\lambda\pi\rho)^2 + \left(t(1-\rho) + \frac{1+\lambda\mathcal{Y}+\lambda\pi\rho\mathcal{H}_\rho[G_{\bullet 0}]}{G_{\rho 0}}\right)^2}\right)$$

(provided it exists, together with eq. for $\lambda\mathcal{Y}$)

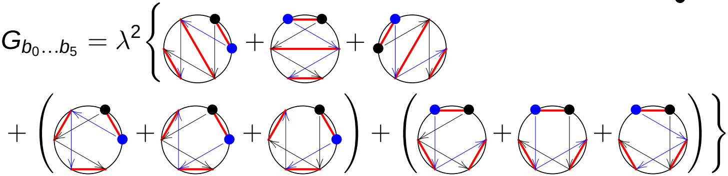
We found nonperturbative results,...

Graphical realisation

$$G_{b_0b_1b_2b_3} = (-\lambda)\frac{G_{b_0b_1}G_{b_2b_3}-G_{b_0b_3}G_{b_2b_1}}{(b_0-b_2)(b_1-b_3)} = -\lambda$$



$$G_{b_0\dots b_5} = \lambda^2$$



b_i ——— b_j $= G_{b_ib_j}$ leads to non-crossing chord diagrams; these are counted by the Catalan number $C_{\frac{N}{2}} = \frac{N!}{(\frac{N}{2}+1)!\,\frac{N}{2}!}$

$b_i \longrightarrow b_j$ $= \frac{1}{b_i-b_j}$ leads to rooted trees connecting the even or odd vertices, intersecting the chords only at vertices

An analogy

2D Ising model	4D nc ϕ^4-theory
temperature T, $K = \frac{J}{k_B T}$	frequency Ω
Kramers-Wannier duality $\sinh(2K)\sinh(2K^*) = 1$	Langmann-Szabo duality $\Omega\Omega^* = 1$
solvable at $K = K^*$ scale-invariant	solvable at $\Omega = \Omega^*$ almost scale-invariant
CFT minimal model ($m = 3$)	matrix model
operator product expansion Virasoro constraints	Schwinger-Dyson equation Ward identities
critical exponents $G^{\sigma\sigma}_{n0} \propto \frac{1}{n^{d-2+\eta}}$, $\eta = \frac{1}{4}$	critical exponents $G^{\phi\phi}_{n0} \propto \frac{1}{n^{1+\lambda}}$, $\lambda \in]0, \lambda_c]$
Virasoro algebra, CFT, subfactors, . . .	**???**

Summary

- We found an exact solution of a Euclidean 4D-quantum field theory. Was unexpected.
- The solution is presumably of little interest for physics. But the mathematical structure is interesting.
- The solved model generalizes the integrable Kontsevich model. Might be relevant for algebraic geometry and combinatorics.
- Expansion of exact solution at $\lambda = 0$ agrees with the Feynman graph computation.
- Motivates looking for alternatives to perturbative QFT in 4D.